Class of Meromorphic Univalent Functions with Fixed Second Positive Coefficients Defined by q-Difference Operator

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Abstract: In this paper using a q-difference operator, a class of meromorphic univalent functions with fixed second positive coefficients is defined. Coefficient estimates, some distortion theorems and other properties for this class are obtained. Various results obtained are sharp.

Keywords: Meromorphic, starlike, convex, fixed coefficient, radius of convexity.

1. INTRODUCTION

For $0 \leq \delta < 1$, let $\Sigma_\delta$ denote the class of univalent meromorphic functions of the form:

$$F(\zeta) = \frac{1}{\zeta^\delta} + \sum_{k=1}^{\infty} a_k \zeta^k, F(\delta) = \infty,$$

defined in the desk $D_\delta = \{\zeta: \delta < |\zeta| < 1\}$. Also let $\Sigma_{\delta, \alpha}$, $0 < \alpha < 1$ be the subclass of functions $F$ in $\Sigma_\delta$ which has the expansion:

$$F(\zeta) = \frac{\alpha}{\zeta^\delta} + \sum_{k=1}^{\infty} a_k \zeta^k,$$

where $\alpha = \text{Res}(F, \delta)$, with $0 < \alpha < 1$, $\zeta \in D_\delta$. The function $F$ given in (1.1) was studied by Jinxi Ma [12]. The functions $F \in \Sigma_\delta$ is said to be meromorphically starlike (convex) functions of order $\beta$ if and only if

$$-\text{Re}\left\{\frac{F'(\zeta)}{F(\zeta)}\right\} \geq \beta, 0 \leq \beta < 1, \zeta \in D_\delta, \quad (1.2)$$

$$-\text{Re}\left\{1 + \frac{F''(\zeta)}{F'(\zeta)}\right\} \geq \beta, 0 \leq \beta < 1, \zeta \in D_\delta. \quad (1.3)$$

The class of such functions is denoted by $\Sigma_{\delta}^*(\beta)$ ($\Sigma_{\delta}^0(\beta)$). Note that the class $\Sigma_{\delta}^0(\beta)$ and various other subclasses of $\Sigma_{\delta}^0(0)$ had been studied by [5](see also [1, 2], [10], [13], [15], [17, 18, 19]).

Let $\Sigma_{\delta, \alpha}^+ \subset \Sigma_{\delta, \alpha}$ consisting of functions of the form:

$$F(\zeta) = \frac{\alpha}{\zeta^\delta} + \sum_{k=1}^{\infty} a_k \zeta^k, (a_k \geq 0). \quad (1.4)$$

It is known that the calculus without the notion of limits is called $q-$calculus which has influenced many scientific fields due to its important applications. Tang et al. [16] defined the $q-$derivative for meromorphic functions $F \in \Sigma_0$ by:

$$\partial_q F(\zeta) = \frac{F(q^m \zeta) - F(q^n \zeta)}{(q^{m-n} - 1)q},$$

where

$$[j]_q = \frac{1-q^j}{1-q}. \quad (1.6)$$

As $q \to 1^-$, $[j]_q = j$ and $\partial_q F(\zeta) = F'(\zeta)$.

For $F \in \Sigma_{\delta, \alpha}$, let:

$$\mathcal{M}_q^0 F(\zeta) = F(\zeta),$$

$$\mathcal{M}_q^1 F(\zeta) = \zeta \partial_q F(\zeta) + \frac{\alpha((q + 1)\zeta - \delta)}{(\zeta - \delta)(q\zeta - \delta)},$$

$$\mathcal{M}_q^2 F(\zeta) = \zeta \partial_q (\mathcal{M}_q^1 F(\zeta)) + \frac{\alpha((q + 1)\zeta - \delta)}{(\zeta - \delta)(q\zeta - \delta)}.$$
and for \( n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) we can write
\[
M_q^n F(\zeta) = \zeta \partial_q \left( M_q^{n-1} F(\zeta) \right) + \frac{\alpha((q + 1)\zeta - \delta)}{(\zeta - \delta)(q\zeta - \delta)} = \frac{\alpha}{\zeta - \delta} + \sum_{k=1}^{\infty} [k]_q^n a_k \zeta^k.
\] (1.7)

Note that:

(i) \( \lim_{q \to 1^-} M_q^n(\delta, \alpha) = M^n(\delta, \alpha) \) (see [7, 8, 9]);

(ii) \( \lim_{q \to 1^-} M_q^n(0, 1) = M^n \) (see [6]).

Using the operator \( M_q^n \), and for \( F \in \Sigma_{\delta, \alpha} \) we have:

**Definition 1** The function \( F \in \Sigma_q^n(\delta, \alpha, \beta) \) if it satisfies
\[
\left| \frac{\zeta \partial_q (M_q^n F(\zeta))}{M_q^n F(\zeta)}' \right| + 1 < \left| \frac{\zeta \partial_q (M_q^n F(\zeta))}{M_q^n F(\zeta)}' \right| + 2 \beta - 1 \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),
\]
(1.8)

for some \( \beta (0 \leq \beta < 1) \).

For \( q \to 1^- \), \( \Sigma_q^n(\delta, 1, \beta) \) is the class of meromorphically starlike functions of order \( \beta \) and \( \Sigma_q^n(\delta, 1, 0) \) gives the meromorphically starlike functions for all \( \zeta \in \mathcal{D}_\delta \).

Note that:

i. \( \lim_{q \to 1^-} \Sigma_q^n(\delta, \alpha, \beta) = \Sigma^n(\delta, \alpha, \beta) \)
ii. \( \Sigma_q^n(0, 1, \beta) = \Sigma^n(\beta) \)
iii. \( \lim_{q \to 1^-} \Sigma_q^n(0, 1, \beta) = \Sigma^n(\beta) \)

\[
\frac{\zeta \partial_q (D^n F(\zeta))}{D^n F(\zeta)}' + 1 < \left| \frac{\zeta \partial_q (D^n F(\zeta))}{D^n F(\zeta)}' \right| + 2 \beta - 1
\]
(1.9)

and for \( e(\zeta) = \frac{1}{\zeta} \);

\[
N_{q, \delta}(e, g) = \{ g : g(\zeta) \in \Sigma_{\delta, \alpha}, g(\zeta) = \frac{1}{\zeta} + \sum_{k=1}^{\infty} b_k \zeta^k \text{ and } \sum_{k=1}^{\infty} k |a_k - b_k| \leq \delta \}.
\] (1.10)

2 MAIN RESULTS

Unless indicated, let \( 0 < q < 1, n \in \mathbb{N}_0, 0 < \alpha \leq 1, 0 \leq \beta < 1, \zeta \in \mathcal{D}_\delta \).

**Theorem 1** Let \( F(\zeta) \) be defined by (1.4). Then \( F \in \Sigma_q^n[\delta, \alpha, \beta] \) if and only if
\[
\sum_{k=1}^{\infty} [k]_q^n \alpha [k]_q + \beta (1 - \delta) a_k \leq \alpha (1 - \beta).
\] (2.1)

**Proof.** Assume that (2.1) holds true and let \( |\zeta| = 1 \), by (1.8) we get
\[
\left| \frac{\zeta \partial_q (M_q^n F(\zeta))}{M_q^n F(\zeta)} + 1 \right| - \left| \frac{\zeta \partial_q (M_q^n F(\zeta))}{M_q^n F(\zeta)} + 2 \beta - 1 \right| \leq \frac{2\alpha(1 - \beta)}{|\zeta||1 - \delta|} - 2 \sum_{k=1}^{\infty} [k]_q^n [q][k]_q + \beta [a_k |\zeta|^{k-1}]
\]
\[
\leq -\frac{2\alpha(1 - \beta)}{1 - \delta} - 2 \sum_{k=1}^{\infty} [k]_q^n [q][k]_q + \beta a_k < 0.
\]

We have
\[
\sum_{k=1}^{\infty} 2[k]_q^n (q[k]_q + \beta)(1 - \delta) a_k - 2\alpha (1 - \beta) < 0
\]

Therefore, by the maximum modules theorem, we have \( F \in \Sigma_q^n [\delta, \alpha, \beta] \).

Now, let \( F \in \Sigma_q^n [\delta, \alpha, \beta] \), then
\[
\left| \frac{c q \partial_q (M_q^n F(\zeta))}{M_q^n F(\zeta)} + 1 \right| < 1, \quad \left| \frac{c q \partial_q (M_q^n F(\zeta))}{M_q^n F(\zeta)} + 2\beta - 1 \right|
\]

since \( \Re(\zeta) \leq |\zeta| \) for all \( \zeta \), we get
\[
\left| c q \partial_q (M_q^n F(\zeta)) \right| < 1, \quad \left| c q \partial_q (M_q^n F(\zeta)) \right| < 1.
\]

Choose values of \( c \) on real axis so that \( \frac{c q \partial_q (M_q^n F(\zeta))}{M_q^n F(\zeta)} \) is real. Letting \( c \to 1^{-} \) through real values, we have (2.1).

**Corollary 1** If \( F \in \Sigma_q^n [\delta, \alpha, \beta] \), then we have
\[
a_k \leq \frac{\alpha(1 - \beta)}{[k]_q^n q[k]_q + \beta(1 - \delta)}. \tag{2.2}
\]

Equality is attained for the function \( F \):
\[
F(\zeta) = \frac{\alpha}{\zeta - \delta} + \frac{\alpha(1 - \beta)}{(q + \beta)(1 - \delta)} \zeta^k. \tag{2.3}
\]

Let \( \Sigma_q^n [\delta, \alpha, \beta, c] \subset \Sigma_q^n [\delta, \alpha, \beta] \) consisting of functions:
\[
F(\zeta) = \frac{\alpha}{\zeta - \delta} + \frac{\alpha(1 - \beta)c}{(q + \beta)(1 - \delta)} \zeta^k + \sum_{k=2}^{\infty} a_k \zeta^k, \tag{2.4}
\]

with \( 0 \leq c < 1 \).

**Theorem 2** Let \( F(\zeta) \) be defined by (2.4). Then \( F(\zeta) \in \Sigma_q^n [\delta, \alpha, \beta, c] \) if and only if
\[
\sum_{k=2}^{\infty} [k]_q^n (q[k]_q + \beta)(1 - \delta) a_k \leq \alpha(1 - \beta)(1 - c), \tag{2.5}
\]

**Proof.** Putting
\[
a_1 = \frac{\alpha(1 - \beta)c}{(q + \beta)(1 - \delta)} \quad (0 < c < 1), \tag{2.6}
\]
in (2.1), we have
\[
c_1 + \sum_{k=2}^{\infty} [k]_q^n (q[k]_q + \beta)(1 - \delta) a_k \leq \frac{\alpha(1 - \beta)c}{c(\delta - \delta)} \leq 1, \tag{2.7}
\]

which implies (2.5). The equality accurs for
\[
F(\zeta) = \frac{\alpha}{\zeta - \delta} + \frac{\alpha(1 - \beta)c}{(q + \beta)(1 - \delta)} \zeta + \frac{\alpha(1 - \beta)(1 - c)}{[k]_q^n (q[k]_q + \beta)(1 - \delta)} \zeta^k, \tag{2.8}
\]

for \( k \geq 2 \).

**Corollary 2** If \( F(\zeta) \in \Sigma_q^n [\delta, \alpha, \beta, c] \), then
\[
a_k \leq \frac{\alpha(1 - \beta)(1 - c)}{[k]_q^n q[k]_q + \beta(1 - \delta)}, \quad (k \geq 2). \tag{2.9}
\]

The equality accurs for \( F(\zeta) \) given by (2.8).

**Theorem 3** Let \( F(\zeta) \in \Sigma_q^n [\delta, \alpha, \beta, c] \), then
\[
\sum_{k=2}^{\infty} a_k \leq \frac{\alpha(1 - \beta)(1 - c)}{[2]_q^n q[2]_q + \beta(1 - \delta)}, \tag{2.10}
\]

and
\[
\sum_{k=2}^{\infty} [k]_q q a_k \leq \frac{[2]_q^n q(1 - \delta) \sum_{k=2}^{\infty} q[k]_q a_k}{[2]_q^n q[2]_q + \beta(1 - \delta)}. \tag{2.11}
\]

**Proof.** Let \( F \in \Sigma_q^n [\delta, \alpha, \beta, c] \). Then, in view of (2.5), we have
\[
[2]_q^n q[2]_q + \beta(1 - \delta) \sum_{k=2}^{\infty} q[k]_q a_k \leq \alpha(1 - \beta)(1 - c), \tag{2.12}
\]

which immediately yields the first assertion.

By appealing to (2.5), we have
\[
[2]_q^n q(1 - \delta) \sum_{k=2}^{\infty} q[k]_q a_k \leq \alpha(1 - \beta)(1 - c) - \beta[2]_q^n q(1 - \delta) \sum_{k=2}^{\infty} q[k]_q a_k, \tag{2.13}
\]

which in view of (2.10), can be putten in the form:
\[
[2]_q^n (1 - \delta) \sum_{k=2}^{\infty} q[k]_q a_k \leq \alpha(1 - \beta)(1 - c) - \beta \frac{\alpha(1 - \beta)(1 - c)}{[2]_q^n q[2]_q + \beta}. \tag{2.14}
\]

Simplifying the right hand side of (2.14), we have (2.11).

**Theorem 4** Let \( F(\zeta) \in \Sigma_q^n [\delta, \alpha, \beta, c] \) for \( 0 < |\zeta| = r < 1 \).

Then
\[
\frac{\alpha}{|\zeta - \delta|} - \frac{\alpha(1 - \beta)c}{(q + \beta)(1 - \delta)} |\zeta| - \frac{\alpha(1 - \beta)(1 - c)}{[2]_q^n q[2]_q + \beta(1 - \delta)} |\zeta|^2
\]
\begin{equation}
\leq |F(\zeta)| \leq \frac{\alpha}{|\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} |\zeta| + \frac{\alpha (1 - \beta) c}{2[2]^n_q(q[2]_q + \beta)(1 - \delta)} |\zeta|^2,
\tag{2.15}
\end{equation}

with equality for
\[F(\zeta) = \frac{\alpha}{|\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} |\zeta| + \frac{\alpha (1 - \beta) c}{2[2]^n_q(q[2]_q + \beta)(1 - \delta)} |\zeta|^2,\]

where \(\alpha = \text{Res}(\zeta, \delta)\), with \(0 < c < 1\).

**Proof.** For \(F(\zeta) \in \Sigma_q^n[\delta, \alpha, \beta, c]\). Then
\[|F(\zeta)| \leq \frac{\alpha}{|\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} |\zeta| + |\zeta|^2 \sum_{k=2}^{\infty} a_k k^k, \]

and
\[|F(\zeta)| \geq \frac{\alpha}{|\zeta - \delta|} - \frac{\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} |\zeta| - |\zeta|^2 \sum_{k=2}^{\infty} a_k k^k, \]

which in view of (2.10), we have (2.15).

**Theorem 5** Let \(F(\zeta) \in \Sigma_q^n[\delta, \alpha, \beta, c]\) for \(0 < |\zeta| = r < 1\), then
\[|\zeta - \delta||q\zeta - \delta| \leq \frac{\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} \frac{\alpha (1 - \beta) c}{\alpha (1 - \beta)(1 - c)} \frac{2\alpha (1 - \beta) c}{|\zeta - \delta||q\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{|\zeta - \delta||q\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{|\zeta - \delta||q\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{2[2]^n_q(q[2]_q + \beta)(1 - \delta)} |\zeta|,
\]

with equality for
\[\partial_q F(\zeta) = \frac{\alpha (1 - \beta) c}{|\zeta - \delta||q\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{2[2]^n_q(q[2]_q + \beta)(1 - \delta)} |\zeta|.
\]

**Proof.** For \(F(\zeta) \in \Sigma_q^n[\delta, \alpha, \beta, c]\). Then
\[|\partial_q F(\zeta)| = \left| -\frac{\alpha}{(q-\delta)(q_\delta-\delta)} + \frac{\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} \right| \sum_{k=2}^{\infty} [k]_q a_k k^{-1} \]

\[\leq \frac{\alpha}{|\zeta - \delta||q\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} + |\zeta| \sum_{k=2}^{\infty} [k]_q a_k k^{-1} \]

\[\geq \frac{\alpha}{|\zeta - \delta||q\zeta - \delta|} - \frac{\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} - |\zeta| \sum_{k=2}^{\infty} [k]_q a_k k^{-1}, \]

which in view of (2.11), we have the result.

**Theorem 6** Let \(F(\zeta) \in \Sigma_q^n[\delta, \alpha, \beta, c]\). Then \(F(\zeta)\) is starlike of order \(\nu\) \((0 \leq \nu < 1)\) in \(|\zeta - \delta| < |\zeta| < r_1\), where \(r_1\) is the largest value for which
\[\alpha (1 - \beta) c \frac{2\alpha (1 - \beta) c}{|\zeta - \delta||q\zeta - \delta|} + \frac{\alpha (1 - \beta) c}{|\zeta - \delta||q\zeta - \delta|} + \sum_{k=2}^{\infty} (k + 1) a_k |\zeta|^k \]

\[\leq \alpha (1 - \nu), \quad \tag{2.16}
\]

for \(k \geq 2\). The result is sharp for the function \(F(\zeta)\) given by (2.8).

**Proof.** It is sufficient to show that
\[\left| \frac{(\nu - \delta) F'(\zeta)}{F(\zeta)} + 1 \right| \leq 1 - \nu, \quad \text{if} \quad |\zeta| < r_1. \quad \tag{2.17}
\]

We have
\[\left| \frac{(\nu - \delta) F'(\zeta)}{F(\zeta)} + 1 \right| \leq \frac{2\alpha (1 - \beta) c}{(q + \beta)(1 - \delta)} + \sum_{k=2}^{\infty} (k + 1) a_k |\zeta|^k \]

\[\leq \alpha (1 - \beta) c \frac{2\alpha (1 - \beta) c}{|\zeta - \delta||q\zeta - \delta|} + \sum_{k=2}^{\infty} (k + 1) a_k |\zeta|^k \]

\[\leq \alpha (1 - \nu), \quad \tag{2.18}
\]

Hence for \(|\zeta - \delta| < |\zeta| < r_1\), (2.18) hold true if
\[\alpha (3 - \nu)(1 - \beta) c \frac{2\alpha (1 - \beta) c}{|\zeta - \delta||q\zeta - \delta|} + \sum_{k=2}^{\infty} (k + 2 - \nu) a_k r^{k+1} \]

\[\leq \alpha (1 - \nu), \quad \tag{2.19}
\]

and it follows that from (2.5), we may take
\[a_k \leq \frac{\alpha (1 - \beta)(1 - c)|\lambda_k|}{|k|_q^n(q[k]_q + \beta)(1 - \delta)}, \quad (k \geq 2),
\]

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where \(\lambda_k \geq 0\) and \(\sum_{k=2}^{\infty} \lambda_k \leq 1\).

For each fixed \(r\), we choose the positive integer \(k_0 = k_0(r)\) for which

\[
\sum_{k=k_0}^{\infty} (k + 2 - \nu) \alpha_k r^{k+1} \leq \frac{\alpha(k_0 + 2 - \nu)(1 - \beta)(1 - c)}{[k_0]^n q[k_0 q + \beta](1 - \delta)} F_{r_0}^{k_0+1},
\]

is maximal.

Then it follows that

\[
\sum_{k=2}^{\infty} (k + 2 - \nu) \alpha_k r^{k+1} \leq \frac{\alpha(k_0 + 2 - \nu)(1 - \beta)(1 - c)}{[k_0]^n q[k_0 q + \beta](1 - \delta)} r^{k_0+1},
\]

then \(F\) is starlike of order \(\nu\) in \(|\zeta - \delta| < |\zeta| < r_1\) provided that

\[
\frac{\alpha(3 - \nu)(1 - \beta)c}{(q + \beta)(1 - \delta)} r_1^2
+ \frac{\alpha(k_0 + 2 - \nu)(1 - \beta)(1 - c)}{[k_0]^n q[k_0 q + \beta](1 - \delta)} r_1^{k_0+1}
\leq \alpha(1 - \nu).
\]

We find the value \(r_1 = r_0(n, \alpha, \beta, c, \nu, k)\) and the corresponding integer \(k_0(r_0)\) so that

\[
\frac{\alpha(3 - \nu)(1 - \beta)c}{(q + \beta)(1 - \delta)} r_0^2
+ \frac{\alpha(k_0 + 2 - \nu)(1 - \beta)(1 - c)}{[k_0]^n q[k_0 q + \beta](1 - \delta)} r_0^{k_0+1}
= \alpha(1 - \nu).
\]

Then this value is the radius of starlikeness of order \(\nu\) for function \(F\) belong to class \(\Sigma_q^n [\delta, \alpha, \beta, c]\).

**Theorem 7** Let \(F(\zeta) \in \Sigma_q^n [\delta, \alpha, \beta, c]\). Then \(F(\zeta)\) is convex of order \(\nu\) \((0 \leq \nu < 1)\) in \(|\zeta - \delta| < |\zeta| < r_2\), where \(r_2\) is the largest value for which

\[
\frac{\alpha(3 - \nu)(1 - \beta)c}{(q + \beta)(1 - \delta)} r_2^2
+ \frac{\alpha(k_0 + 2 - \nu)(1 - \beta)(1 - c)}{[k_0]^n q[k_0 q + \beta](1 - \delta)} r_2^{k_0+1}
\leq \alpha(1 - \nu),
\]

for \(k \geq 2\). The result is sharp for the function \(F(\zeta)\) given by (2.8).

**Proof.** By using the same technique in the proof of Theorem 6 we can show that

\[
\left|\frac{(c-\delta)F''(\zeta)}{F'(\zeta)} + 2\right| \leq 1 - \nu, \quad (|\zeta| < r_2),
\]

for \(|\zeta - \delta| < |\zeta| < r_2\) with the aid of Theorem 2. Thus, we have the assertion of Theorem 7.

**Theorem 8** The class \(\Sigma_q^n [\delta, \alpha, \beta, c]\) is closed under convex linear combination.

**Proof.** Let \(F(\zeta)\) be defined by (2.4). Define the function \(h(\zeta)\) by

\[
h(\zeta) = \frac{\alpha}{\zeta - \delta} + \frac{\alpha(1 - \beta)c}{(q + \beta)(1 - \delta)} \zeta + \sum_{k=2}^{\infty} b_k \zeta^k, \quad b_k \geq 2.
\]

Suppose that \(F(\zeta)\) and \(h(\zeta)\) are in the class \(\Sigma_q^n [\delta, \alpha, \beta, c]\), we only need to prove that

\[
G(\zeta) = \zeta F(\zeta) + (1 - \zeta) h(\zeta) \quad (0 \leq \zeta \leq 1),
\]

also be in the class. Since

\[
G(\zeta) = \frac{\alpha}{\zeta - \delta} + \frac{\alpha(1 - \beta)c}{(q + \beta)(1 - \delta)} \zeta + \sum_{k=2}^{\infty} \left[\zeta a_k + (1 - \zeta) b_k \right] \zeta^k,
\]

then

\[
\sum_{k=2}^{\infty} [k]^n q[k q + \beta](1 - \delta) \left[\zeta a_k + (1 - \zeta) b_k \right] \zeta^k \leq \alpha(1 - \beta)(1 - c),
\]

with the aid of Theorem 2. Hence \(G(\zeta) \in \Sigma_q^n [\delta, \alpha, \beta, c]\).

**Theorem 9** Let

\[
F_1(\zeta) = \frac{\alpha}{\zeta - \delta} + \frac{\alpha(1 - \beta) c}{(q + \beta)(1 - \delta)} \zeta,
\]

and

\[
F_k(\zeta) = \frac{\alpha}{\zeta - \delta} + \frac{\alpha(1 - \beta) c}{(q + \beta)(1 - \delta)} \zeta + \frac{\alpha(1 - \beta)(1 - c)}{[k]^n q[k q + \beta](1 - \delta)} \zeta^k,
\]

for \(k \geq 2\). Then \(F(\zeta) \in \Sigma_q^n [\delta, \alpha, \beta, c]\) iff

\[
F(\zeta) = \sum_{k=2}^{\infty} \eta_k F_k(\zeta),
\]

where \(\eta_k \geq 0 \quad (k \geq 2)\) and

\[
\sum_{k=2}^{\infty} \eta_k \leq 1.
\]

**Proof.** Let \(F(\zeta)\) be in the form (2.27). Then from (2.25), (2.26) and (2.28) we have
\[ F(\zeta) = \frac{\alpha}{\zeta - \delta} + \frac{\alpha(1 - \beta)c}{(q + \beta)(1 - \delta)} \zeta + \sum_{k=2}^{\infty} \frac{\alpha(1 - \beta)(1 - c)\eta_k}{[k]_q^p(q[k]_q + \beta)(1 - \delta)} k^k \zeta^k. \]  

(2.29)

Since

\[ \sum_{k=2}^{\infty} \frac{\alpha(1 - \beta)(1 - c)\eta_k}{[k]_q^p(q[k]_q + \beta)(1 - \delta)} = \sum_{k=2}^{\infty} \eta_k = 1 - \eta_1 \leq 1, \]  

(2.30)

then, from Theorem 2, \( F(\zeta) \in \Sigma^n_q[\delta, \alpha, \beta, c] \).

Conversely, let \( F(\zeta) \in \Sigma^n_q[\delta, \alpha, \beta, c] \) and satisfies (2.9) for \( k \geq 2 \), then

\[ \eta_k = \frac{[k]_q^p(q[k]_q + \beta)(1 - \delta)}{\alpha(1 - \beta)(1 - c)} a_k \leq 1, \]  

(2.31)

and

\[ \eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k. \]  

(2.32)

This completes the proof of the Theorem 9.

**Corollary 3** The extreme points of the class \( \Sigma^n_q[\delta, \alpha, \beta, c] \) are the functions \( F_k(\zeta) \) \((k \geq 2)\) given by (2.25) and (2.26).

**Theorem 10** If \( F(\zeta) \in \Sigma^n_q[\delta, \alpha, \beta, c] \), then

\[ \Sigma^n_q[\delta, \alpha, \beta, c] \subset N_{q, \xi}(F; q), \]  

(2.33)

where the parameter \( \xi_q \) is given by

\[ \xi_q = \frac{[2]_q^p(1 - \beta)(1 - \alpha)}{[2]_q^p(1 - \alpha)}, \]  

(2.34)

**Proof.** For \( F(\zeta) \in \Sigma^n_q[\delta, \alpha, \beta, c] \), from (2.11) of Theorem 3 and in view of (1.10), we get (2.34).

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4. **REFERENCES**

